§4.3 The solution of the Renormalization Group equation

Recall from last time:

$$\left[\kappa\frac{\partial}{\partial\kappa} + \beta(u)\frac{\partial}{\partial u} - \frac{1}{2}N\gamma_\phi(u)\right]\Gamma_R^{(N)}(k_i;u,\kappa) = 0 \qquad (1)$$

where

$$\beta(u) = \left(\kappa\frac{\partial u}{\partial\kappa}\right)_\lambda, \quad \gamma_\phi(u) = \kappa\left(\frac{\partial \ln Z_\phi}{\partial\kappa}\right)_\lambda$$

Claim:

$$\Gamma_R^{(N)}(k_i;u,\kappa) \equiv \exp\left[-\frac{N}{2}\int_1^\rho \gamma_\phi(u(\rho))\frac{dx}{x}\right]\Gamma_R^{(N)}(k_i,u(\rho),\kappa\rho) \qquad (2)$$

is a solution of (1).

Here $u(\rho)$ is the solution of the diff. equation:

$$\rho\frac{\partial u(\rho)}{\partial\rho} = \beta(u(\rho)), \quad u(\rho=1) = 1 \qquad (3)$$

Proof:

$$\left[\kappa\frac{\partial}{\partial\kappa} + \beta(u)\frac{\partial}{\partial u} - \frac{1}{2}N\gamma_\phi(u)\right]\Gamma_R^{(N)}(k_i;u,\kappa)$$

$$= \exp\left[-\frac{N}{2}\int_1^\rho \gamma_\phi(u(x))\frac{dx}{x}\right]\left[\kappa\rho\frac{\partial}{\partial(\kappa\rho)} + \beta(u)\frac{\partial}{\partial u}\left(-\frac{N}{2}\int_1^\rho \gamma_\phi(u(x))\frac{dx}{x}\right)\right.$$

$$\left. + \beta(u)\frac{\partial u(\rho)}{\partial u}\frac{\partial}{\partial u(\rho)} - \frac{1}{2}N\gamma_\phi(u)\right]\Gamma_R^{(N)}(k_i;u(\rho),\kappa\rho)$$

Use

$$-\frac{N}{2}\int_1^{\rho} \gamma_\phi(u(x))\,\frac{dx}{x} \overset{(3)}{=} -\frac{N}{2}\int_u^{u(\rho)} \frac{\gamma_\phi(u)}{\beta(u)}\,du$$

$$\Rightarrow \beta(u)\frac{\partial}{\partial u}\left(-\frac{N}{2}\int_1^{\rho}\gamma_\phi(u(x))\frac{dx}{x}\right)$$

$$= \beta(u)\left(-\frac{N}{2}\frac{\partial u(\rho)}{\partial u}\frac{\gamma_\phi(u(\rho))}{\beta(u(\rho))} + \frac{N}{2}\frac{\gamma_\phi(u)}{\beta(u)}\right)$$

$$= -\frac{N}{2}\underbrace{\beta(u)\frac{\partial u(\rho)}{\partial u}}_{\substack{= \kappa\frac{\partial u}{\partial \kappa}\frac{\partial u(\rho)}{\partial u} \\ = \kappa\frac{\partial u(\rho)}{\partial \kappa} = \kappa\rho\frac{\partial u(\rho)}{\partial(\kappa\rho)} = \beta(u(\rho))}}\frac{\gamma_\phi(u(\rho))}{\beta(u(\rho))} + \frac{N}{2}\gamma_\phi(u)$$

$$= -\frac{N}{2}\gamma_\phi(u(\rho)) + \frac{N}{2}\gamma_\phi(u)$$

$$= \exp\left[-\frac{N}{2}\int_1^{\rho}\gamma_\phi(u(x))\frac{dx}{x}\right]\underbrace{\left[(\kappa\rho)\frac{\partial}{\partial(\kappa\rho)} + \beta(u(\rho))\frac{\partial}{\partial u(\rho)} - \frac{N}{2}\gamma_\phi(u(\rho))\right]\Gamma_R^{(N)}(k_i, u(\rho), \kappa\rho)}_{=0}$$

$$= 0 \qquad \square$$

In Chapter 3 we had seen that

$$[\Gamma^{(N)}] = \Lambda^{N+d-\frac{1}{2}Nd}$$

$\longrightarrow$ rescaling gives (4)

$$\Gamma_R^{(N)}(k_i; u, \kappa) = (\kappa\rho)^{(N+d-\frac{1}{2}Nd)}\exp\left[-\frac{N}{2}\int\gamma_\phi(u(x))\frac{dx}{x}\right]\Gamma_R^{(N)}\left(\frac{k_i}{\kappa\rho}; u(\rho), 1\right)$$

Now replace k_i by ρk_i, giving:

$$\Gamma_R^{(N)}(\rho k_i; u, \kappa) = \rho^{N+d-\frac{1}{2}Nd} \exp\left[-\frac{N}{2}\int_1^{\rho} \gamma_\phi(u(x))\frac{dx}{x}\right] \times \Gamma_R^{(N)}(k_i; u(\rho), \kappa) \qquad (5)$$

$\rightarrow$ under rescaling of momenta we get:

a) multiplication by that scale to the canonical dimension

b) a modified coupling constant, eq. (3)

c) an additional complicated factor

Intuitive picture for eq. (1):

one-dim. flow of particles in a fluid ($t = \ln \kappa$ is time, u is space-coordinate, $\beta(u)$ is velocity of fluid, $\frac{1}{2}N\gamma_\phi$ is source or sink term)

Let's specify eq. (4) to $\Gamma^{(2)}$:

$$\Gamma_R^{(2)}(k; u, \kappa) = (\kappa\rho)\exp\left[-\int_1^{\rho}\gamma_\phi \frac{dx}{x}\right]\Gamma_R^{(2)}\left(\frac{k}{\kappa\rho}; u(\rho), 1\right)$$

Choose $\rho = \frac{k}{\kappa}$

$$\rightarrow \Gamma_R^{(2)}(k; u, \kappa) = k^2 \exp\left[-\int_1^{k/\kappa}\gamma_\phi \frac{dx}{x}\right]\Phi(u(k/\kappa))$$

where $\Phi = \Gamma_R^{(2)}(1; u, 1)$

§4.4 Fixed points, scaling, and anomalous dimensions

critical values of coupling constant:

$$\beta(u) = 0$$

(3) $\longrightarrow$ u becomes "stationary" or a "fixed point"

eq. (1) at a fixed point $u = u^*$ becomes:

$$\left[\kappa \frac{\partial}{\partial \kappa} - \frac{1}{2} N \gamma_\phi(u^*)\right] \Gamma_R^{(N)}(k_i; u^*, \kappa) = 0$$

$\longrightarrow$ solution is:

$$\Gamma_R^{(N)}(k_i; u^*, \kappa) = \kappa^{\frac{1}{2} N \gamma_\phi(u^*)} \, \Phi(k_i)$$

Then eq. (5) gives

$$\Gamma_R^{(N)}(\rho k_i; u^*, \kappa) = \rho^{(N + d - \frac{1}{2} N d) - \frac{1}{2} N \gamma_\phi(u^*)} \, \Gamma_R^{(N)}(k_i; u^*, \kappa) \qquad (6)$$

$\longrightarrow$ simple scaling behaviour

$\Gamma_R^{(2)}$ behaves as

$$\Gamma_R^{(2)}(\rho k) = \rho^{2 - \gamma_\phi(u^*)} \, \Gamma(k)$$

$\eta \equiv \frac{1}{2} \gamma_\phi(u^*)$ is called the "anomalous dimension" of the field ϕ.

Let us explain this:

Since Γ has dimension $\Lambda^{N+d-\frac{1}{2}Nd}$,

the free vertex scales as

$$\Gamma^{(N)0}(\rho k_i) = \rho^{d-N[(d/2)-1]}\,\Gamma^{(N)0}(k_i)$$

$\longrightarrow$ define the dimension of ϕ by:

$$\Gamma^{(N)}(\rho k_i) = \rho^{d-Nd_\phi}\,\Gamma^{(N)}(k_i)$$

$\longrightarrow$ in the free theory we get

$$d_\phi^0 = \frac{d}{2} - 1 ,$$

i.e the naive dimension of the field.

Now at a fixed point, we have the scaling (6):

$$d_\phi = \frac{d}{2} - 1 + \eta , \quad \eta \neq 0$$

In such a situation, we have

$$\Gamma_R^{(2)}(k) = C\,\kappa^{2\eta}\,k^{2-2\eta} , \quad C \text{ constant}$$

$\longrightarrow$ dimensional analysis gives $\Gamma^{(2)}(k) = C'\Lambda^{2\eta}k^{2-2\eta}$

$\longrightarrow$ at a fixed point $u = u^*$:

$$Z_\phi(u^*, \kappa/\Lambda) = C''(\kappa/\Lambda)^{\eta}$$

Approaching the fixed point - asymptotic freedom:

Assume that β has simple zero at u^*

$\rightarrow \beta(u) = a(u^* - u)$

Inserting into (3), we get

$$\frac{\partial u(s)}{\partial s} = a(u^* - u), \quad \text{where } s \equiv \ln \rho$$

$\rightarrow$ solved by $u(s) = u^* - c\, e^{-as}$ (7)

boundary condition: $u(s=0) = u_0 \Rightarrow u^* - c = u_0$
and $c = u^* - u_0$ (8)

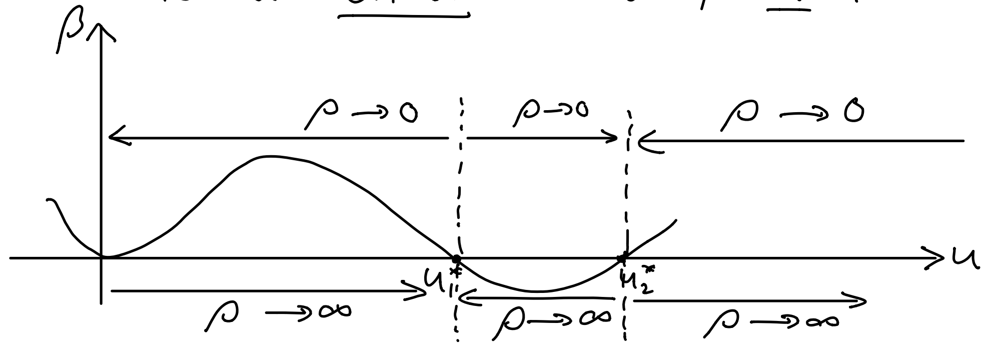
Two limits of relevance:

- $s \rightarrow -\infty$, i.e. $\rho \rightarrow 0$: relevant for IR physics
- $s \rightarrow +\infty$, i.e. $\rho \rightarrow \infty$: relevant for UV physics

From (7) we see:

- if $a > 0$: $u(s) \xrightarrow{s \rightarrow \infty} u^*$
- if $a < 0$: $u(s) \xrightarrow{s \rightarrow -\infty} u^*$

$\rightarrow$ Renormalized coupling constant will flow to u^* <u>either</u> in UV, <u>or</u> in IR!



We call u_1^* a UV-stable fixed point and u_2^* an IR-stable fixed point.
If we start at a UV fixed-point and $\rho \to 0$, then u will move away from that point towards the nearest IR-fixed point. $\longrightarrow$ coupling constants "attracted" to the IR fixed-point

Let us see what happens to γ_ϕ:

$$\gamma_\phi(u) = \gamma_\phi^* + \gamma_\phi'(u-u^*)$$

$$\rightarrow \int_1^\rho \gamma_\phi(u(x))\,\frac{dx}{x} = \int_{u_0}^{u(s)} \frac{\gamma_\phi(u')}{\beta(u')}\,du'$$

$$= \gamma_\phi^* \int_{u_0}^{u(s)} \frac{du'}{a(u^*-u')} - \frac{\gamma_\phi'}{a}\int_{u_0}^{u(s)} du'$$

$$= \frac{\gamma_\phi^*}{a}\underbrace{\left[\ln(u^*-u_0) - \ln(\underbrace{u^*-u(s)}_{\stackrel{(8)}{=}(u^*-u_0)e^{-as}})\right]}_{=-\ln(e^{-as})} - \frac{\gamma_\phi'}{a}\left[u(s)-u_0\right]$$

$$= \gamma_\phi^* s - \frac{\gamma_\phi'}{a}\left[u(s)-u_0\right] \qquad (9)$$

$$\stackrel{u(s)\to u^*}{\Longrightarrow} \exp\left[-\frac{N}{2}\int_1^\rho \gamma_\phi\,\frac{dx}{x}\right] = \underbrace{e^{-\frac{N}{2}\gamma_\phi^* s}}_{=\rho^{-N\gamma_\phi^*/2}}\ \underbrace{e^{\frac{N}{2}\frac{\gamma_\phi'}{a}(u^*-u_0)}}_{\equiv C} = C\rho^{-N\gamma_\phi^*/2} \qquad (10)$$

→ obtain anomalous dimension as asymptotic behavior of vertex functions as functions of the scale ρ of the momenta!

Now consider the following situation:

- start at an interacting theory, $u_0 \neq 0$
- suppose β vanishes as $u \to 0$

→ in the asymptotic limits (UV or IR), $u(\rho)$ of tend to zero

→ theory will be "asymptotically free"